Complex Numbers

- Let $Z = a + jb$ be a complex number
- a is known as real part and b as imaginary part
- j is defined as $j = \sqrt{-1}$
- Complex conjugate of Z is $\overline{Z} = a - jb$
- Modulus of complex number is defined as $|Z| = \sqrt{Z\overline{Z}} = \sqrt{a^2 + b^2}$

Polar Representation of Complex Numbers

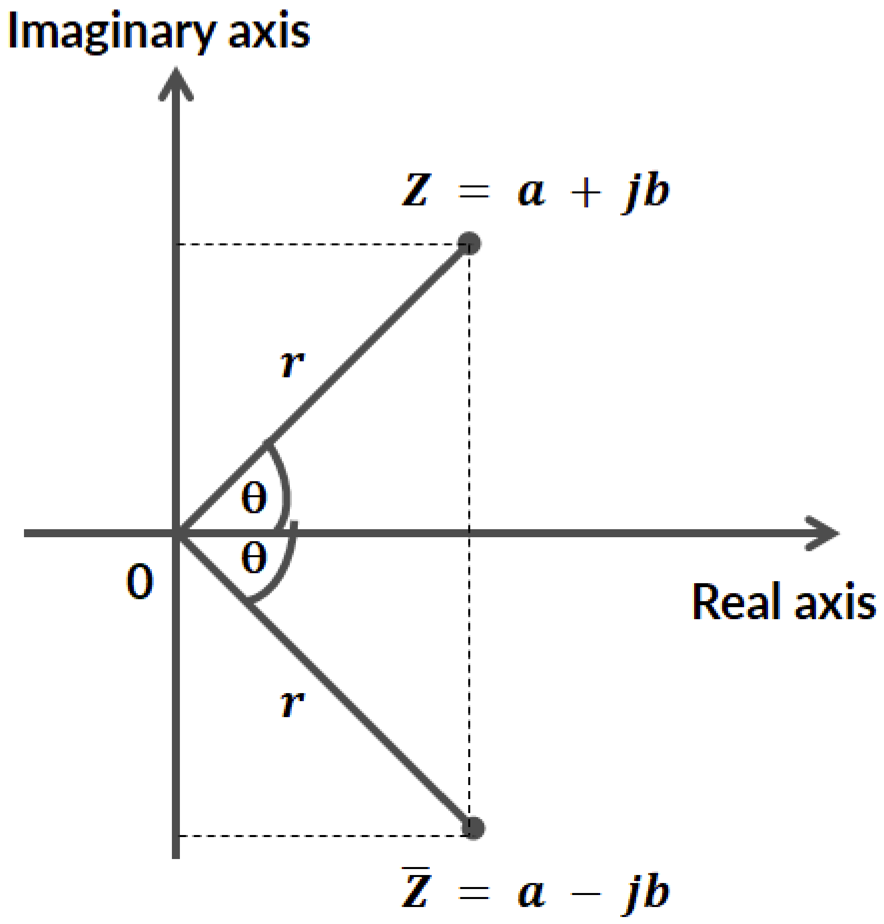
- A complex number *Z=a+jb* can be represented in a polar form
- In polar form, $Z = a + jb = re^{j\theta}$
- where $r = |Z| = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

$$\overline{Z} = a - jb = re^{-j\theta}$$

$$|Z| = \sqrt{Z\overline{Z}} = \sqrt{re^{j\theta} re^{-j\theta}} = \sqrt{r^2} = r$$

- If we know r and θ, we can obtain a and b as

$$a = r\cos\theta \quad \text{and} \quad b = r\sin\theta$$



Arithmetic Operations of Complex Numbers: Cartesian Representation

Let $z_1 = a_1 + jb_1$ and $z_2 = a_2 + jb_2$

Addition: $z_1 + z_2 = (a_1 + a_2) + j(b_1 + b_2)$

Subtraction: $z_1 - z_2 = (a_1 - a_2) + j(b_1 - b_2)$

Multiplication: $z_1 \times z_2 = (a_1 a_2 - b_1 b_2) + j(b_1 a_2 + b_2 a_1)$

Division:

$$\frac{z_1}{z_2} = \frac{(a_1 a_2 + b_1 b_2) + j(b_1 a_2 - b_2 a_1)}{{a_2}^2 + {b_2}^2}$$

Note: It is easy to add or subtract complex number in Cartesian representation

Arithmetic Operations of Complex Numbers: Polar Representation

Let $z_1 = r_1 e^{j\theta_1}$ and $z_2 = r_2 e^{j\theta_2}$

Multiplication:

$$z_1 \times z_2 = r_1 r_2 e^{j(\theta_1 + \theta_2)}$$

Division:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)}$$

Note: It is easy to multiply or divide complex number in Polar representation

- The idea of phasor representation is based on Euler's identity

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta$$

$$\cos\theta = \mathrm{Re}\left(e^{j\theta}\right) \qquad \sin\theta = \mathrm{Im}\left(e^{j\theta}\right)$$

Capacitors

- A capacitor is a passive element designed to store energy in its electric field.

- Capacitors are used extensively in electronics, communications, computers, and power systems. For example, they are used in the tuning circuits of radio receivers and as dynamic memory elements in computer systems.

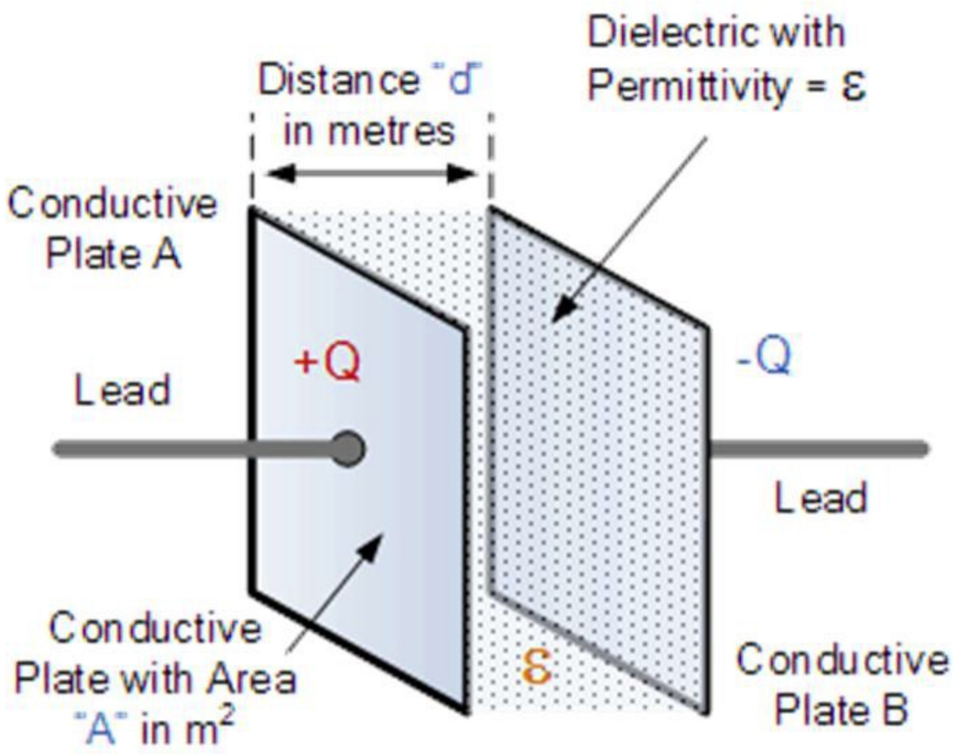
- Two conductive plates separated by an insulator (or dielectric) forms a capacitor. Commonly illustrated as two parallel metal plates separated by a distance, d.

$$C = \frac{\varepsilon A}{d}$$

where $\varepsilon = \varepsilon_0 \varepsilon_r$

ε_r is the relative dielectric constant

ε_0 is the vacuum permittivity



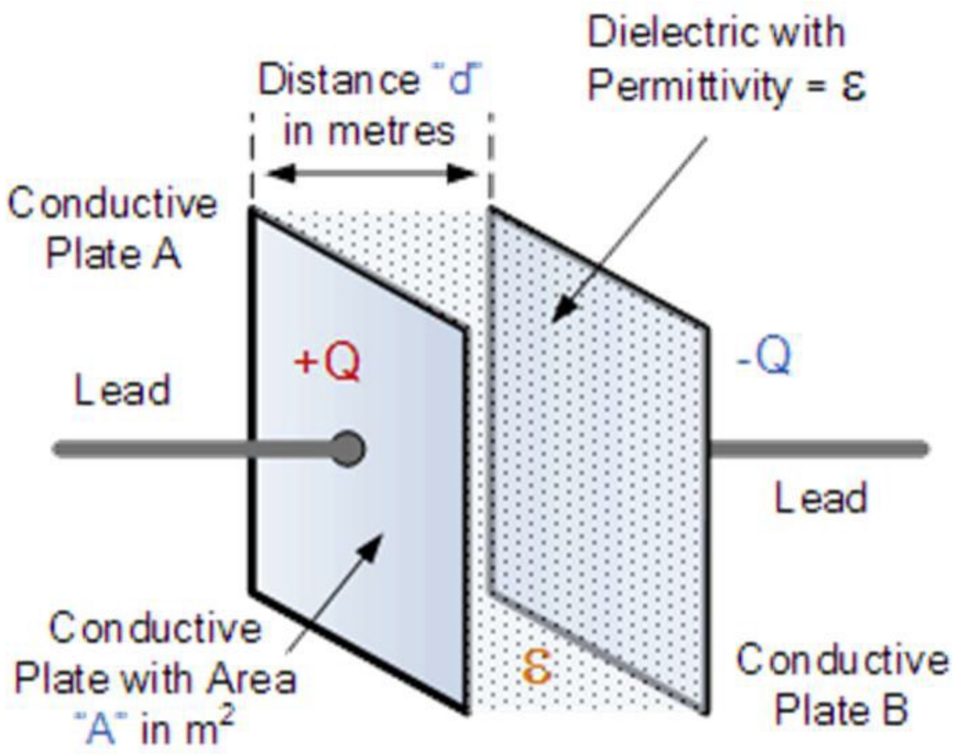
Effect of Dimensions on Capacitors

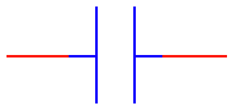
Capacitance increases with

- increasing surface area of the plates,
- decreasing spacing between plates
- increasing the relative dielectric constant of the insulator between the two plates.

$$C = \frac{\varepsilon A}{d}$$

Typically, capacitors have values in the picofarad (pF) to microfarad (μF) range.





Capacitor Circuit Symbol

Types of Capacitors

Fixed Capacitors

– Nonpolarized

- May be connected into circuit with either terminal of capacitor connected to the high voltage side of the circuit.
- Insulator: Paper, Mica, Ceramic, Polymer

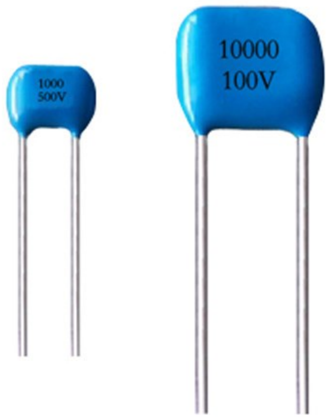
– Polarized

Electrolytic

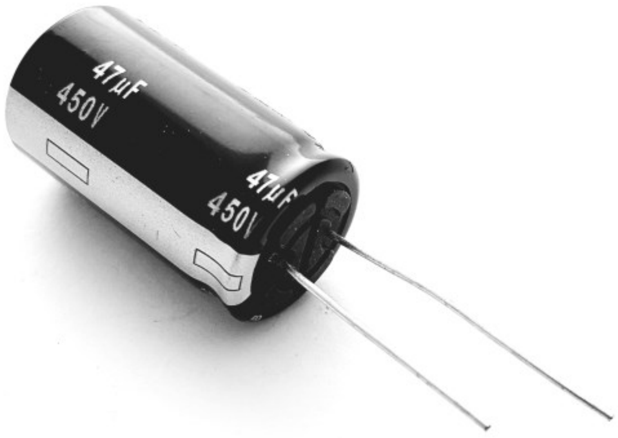
- The negative terminal must always be at a lower voltage than the positive terminal
- Plates or Electrodes: Aluminum, Tantalum

- Difficult to make nonpolarized capacitors that store a large amount of charge or operate at high voltages.

- Tolerance on capacitance values is very large: can be as high as ±20%



Mica Capacitor

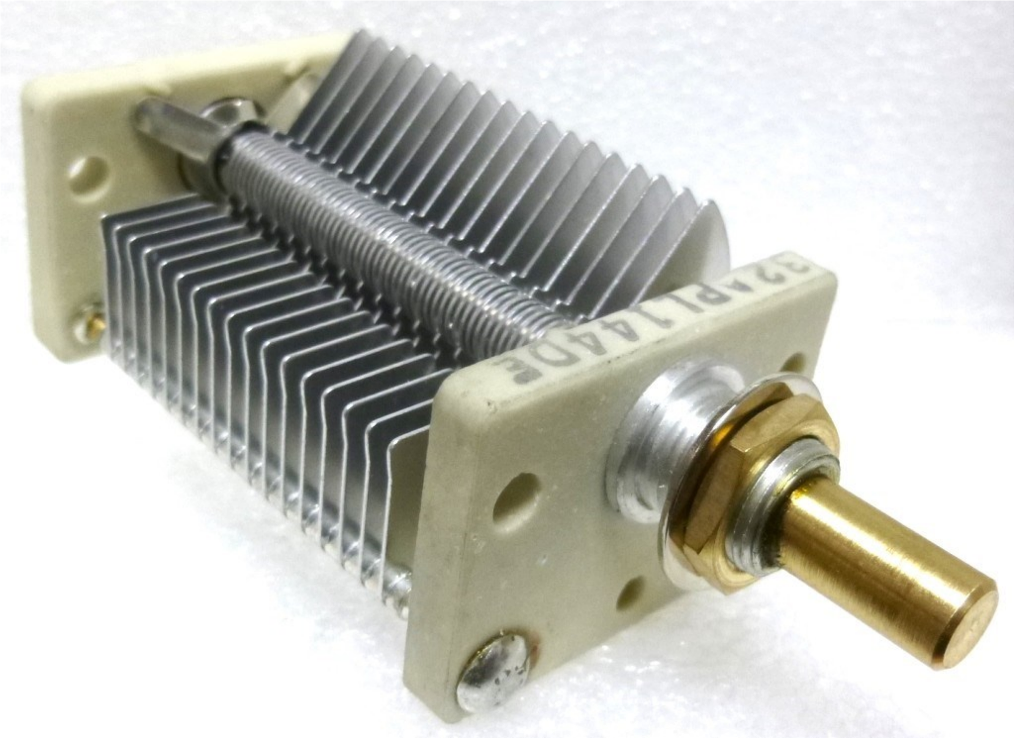


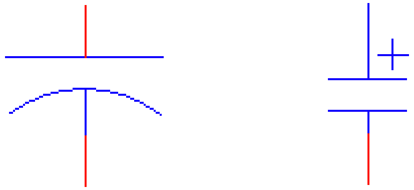
Electrolytic Capacitor

Types of Capacitors…Continued

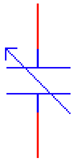
Variable Capacitors

- Cross-section area of capacitor plate is changed as one set of plates are rotated with respect to the other.





Polarized Capacitor Symbol



Variable Capacitor Symbol

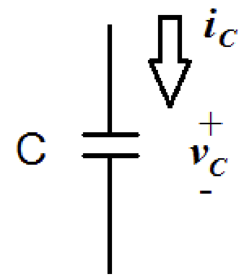
https://www.rfparts.com/review/product/list/id/13998/category/1453/

Electrical Properties of a Capacitor

- Acts like an open circuit at steady state when connected to a d.c. voltage or current source.
- Voltage on a capacitor must be continuous. There are no abrupt changes to the voltage, but there may be discontinuities in the current.
- An ideal capacitor does not dissipate energy, it uses power when charging energy and returns power when discharging energy.

Sign Conventions

The sign convention used with a capacitor is the same as for a power dissipating device.



- When current flows into the positive side of the voltage across the capacitor, it is positive, and the capacitor is dissipating power.
- When the capacitor releases energy back into the circuit, the sign of the current will be negative.

Current-Voltage Relationships

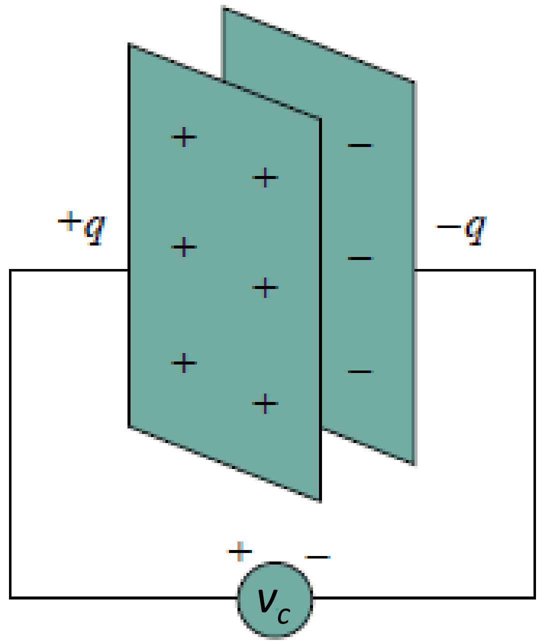
When a voltage source v_c is applied across a capacitor, a positive charge $+q$ is stored on one plate and a negative charge $-q$ on the other. The amount of charge stored, represented by q, is directly proportional to the applied voltage v_C.

$$q = Cv_C$$

$$i_C = \frac{dq}{dt}$$

$$i_C = C\frac{dv_C}{dt}$$

$$v_C = \frac{1}{C}\int_{t_0}^{t_1} i_C dt$$



Current-Voltage Relationships…Continued

Let $$v_C = V_0 \sin \omega t$$

$$i_C = C \frac{d}{dt}\left(V_0 \sin \omega t\right)$$

$$i_C = \omega C V_0 \cos \omega t$$

$$i_C = \omega C V_0 \sin\left(\frac{\pi}{2} - \omega t\right)$$

Current through capacitor leads voltage across capacitor by $\frac{\pi}{2}$

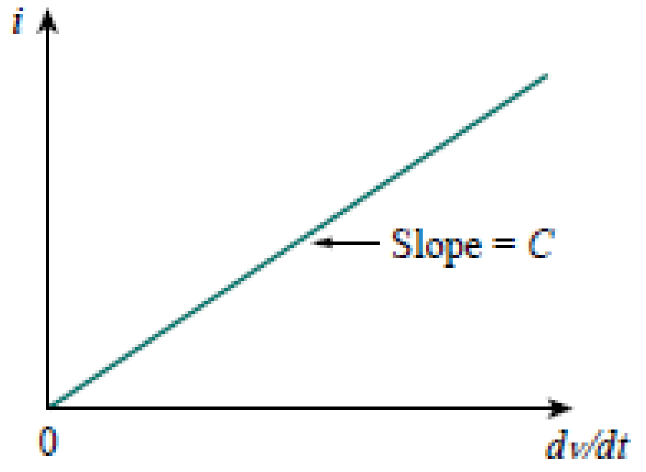
$$i_C = C\frac{dv_C}{dt}$$

Let

$$v_C = V_0 e^{j\omega t}$$

$$i_C = C\frac{d}{dt}\left(V_0 e^{j\omega t}\right) = j\omega C V_0 e^{j\omega t}$$

$$v_C = \frac{1}{j\omega C} i_C$$



The current-voltage relationship is illustrated in the Figure for a capacitor whose capacitance is independent of voltage. Capacitors that satisfy Equation $i_C = C\frac{dv_C}{dt}$ are said to be *linear.* For a *nonlinear capacitor*, the plot of the current-voltage relationship is not a straight line.

Reactance and Impedance of a Capacitor

$$v_C = \frac{1}{j\omega C} i_C = -jX_C i_C = Z_C i_C$$

$$X_C = \frac{1}{\omega C}$$

$$Z_C = \frac{1}{j\omega C}$$

- X_C is called the reactance of the capacitor and Z_C, the impedance of the capacitor.

– When ω= 0, X_C = ∞, means reactance is infinite i.e., Capacitor blocks DC

– When ω = ∞, X_C = 0, means capacitor behaves like a short at higher frequencies

– Frequency dependent electrical behavior of capacitance on circuit

Power and Energy

Charge is stored on the plates of the capacitor.

Equation: Q = CV

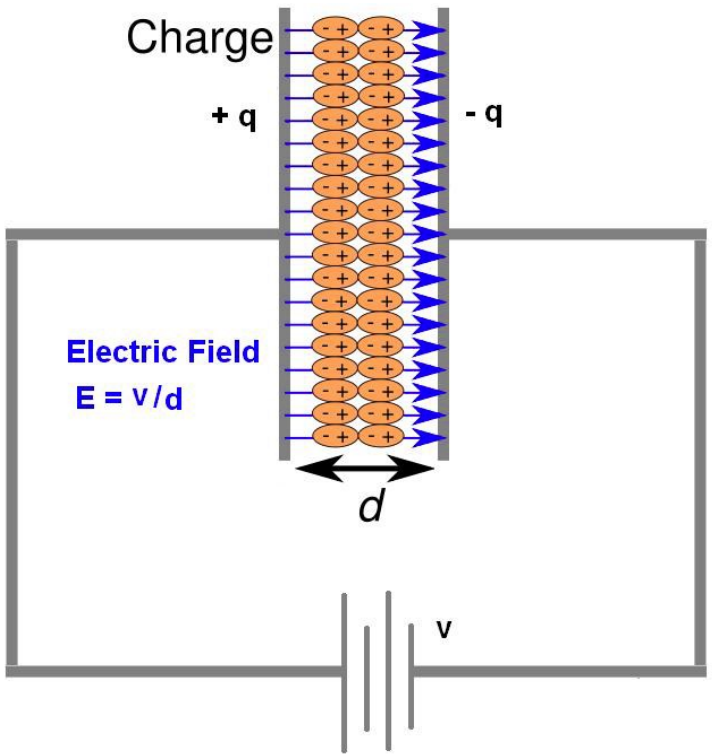
Units:
Farad = Coulomb/Voltage; Farad is abbreviated as F

- A capacitor stores energy when charged and gives away energy while discharging.
- The instantaneous power delivered to the capacitor is

$$p_c = v_c i_c = C v_c \frac{dv_c}{dt}$$

- The energy stored in the capacitor is therefore

$$w_C = \int_{-\infty}^{t} p_C dt = C \int_{-\infty}^{t} v_c \frac{dv_c}{dt} dt = C \int_{-\infty}^{t} v_c dv_c = \frac{1}{2} C v_C^2$$

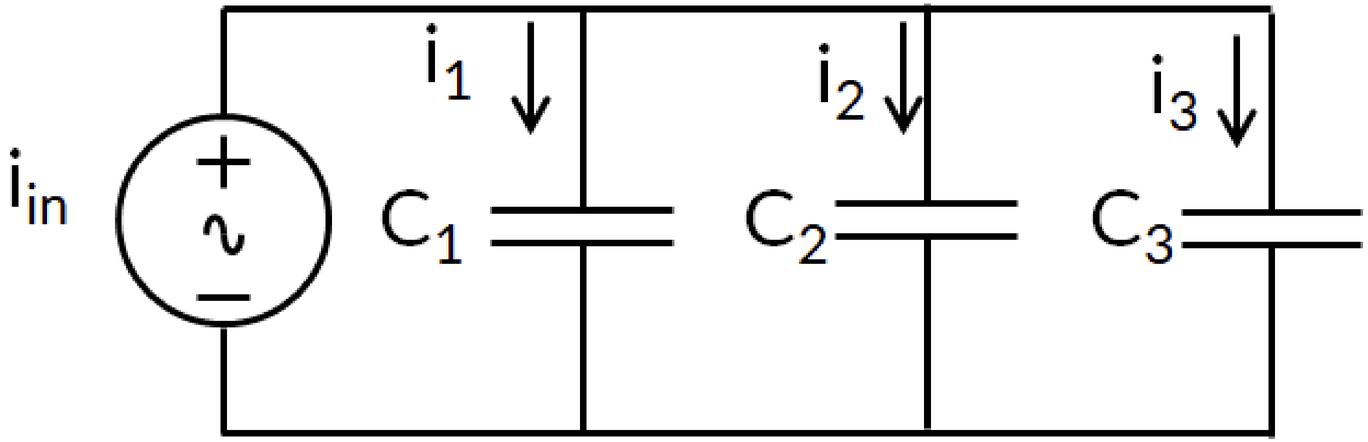


Capacitors in Parallel

- Consider capacitors connected in parallel configuration

– Voltage across the capacitors is equal

– Current is different



Writing KCL,

$$i_{in} = i_1 + i_2 + i_3$$

Noting the current voltage relation for a capacitor as

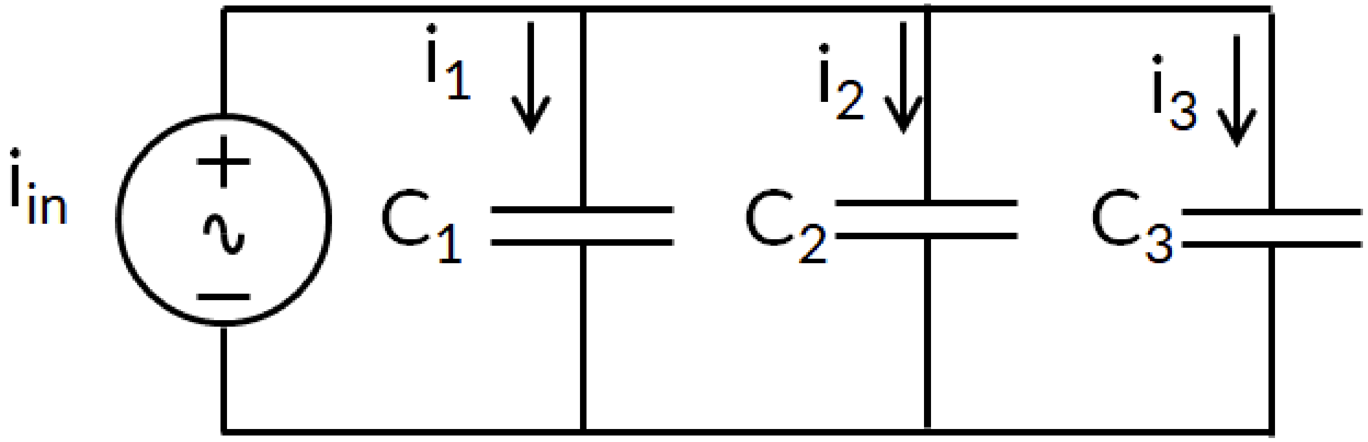
$$i = C\frac{dv}{dt}$$

If C_{eq} is the net capacitance, then $i_C = C_{eq}\frac{dv}{dt}$

Capacitors in Parallel

Writing current voltage relations for individual capacitances as

$$i_1 = C_1 \frac{dv}{dt} \qquad i_2 = C_2 \frac{dv}{dt} \qquad i_3 = C_3 \frac{dv}{dt}$$



Substituting into KCL,

$$i_{in} = C_1 \frac{dv}{dt} + C_2 \frac{dv}{dt} + C_3 \frac{dv}{dt}$$

Hence,

$$C_{eq} = C_1 + C_2 + C_3$$

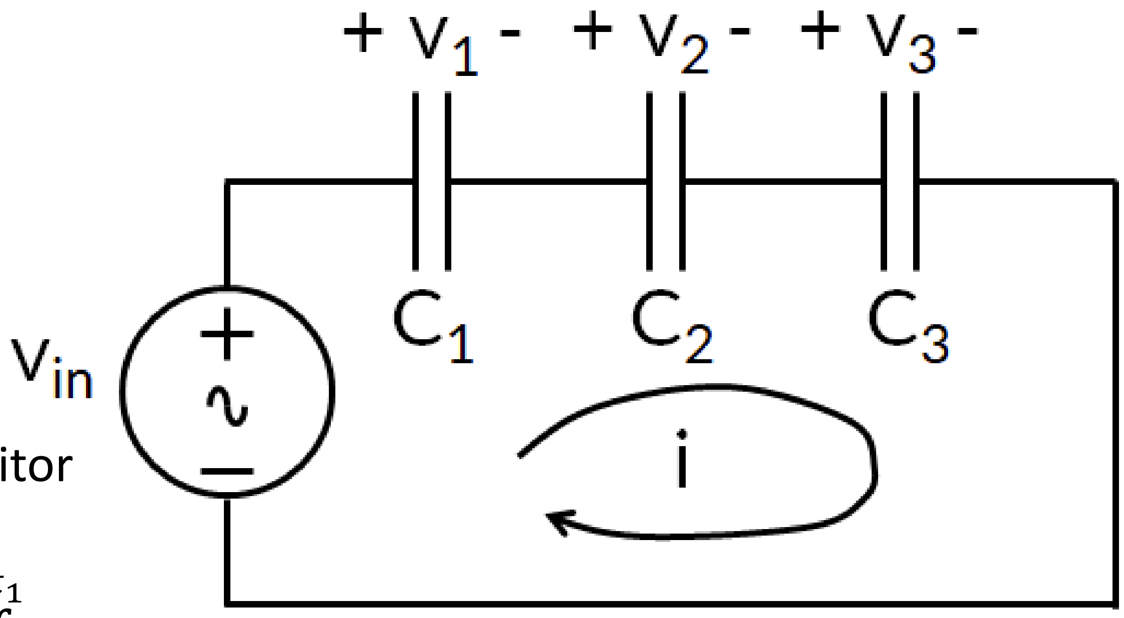
$$\boxed{C_{eq} = \sum_{p=1}^{m} C_p}$$

Capacitors in Series

- Consider capacitors connected in series configuration

– Current through the capacitors is same

– Voltage divides



Writing KVL

$$v_{in} = v_1 + v_2 + v_3$$

- Noting the relation between voltage across the capacitor and current through a capacitor as

$$v_1 = \frac{1}{C_1}\int_{t_0}^{t_1} idt \qquad v_2 = \frac{1}{C_2}\int_{t_0}^{t_1} idt \qquad v_3 = \frac{1}{C_3}\int_{t_0}^{t_1} idt$$

- If C_{eq} is the total capacitance, then

$$v_{in} = \frac{1}{C_{eq}}\int_{t_0}^{t_1} idt$$

Capacitors in Series…Continued

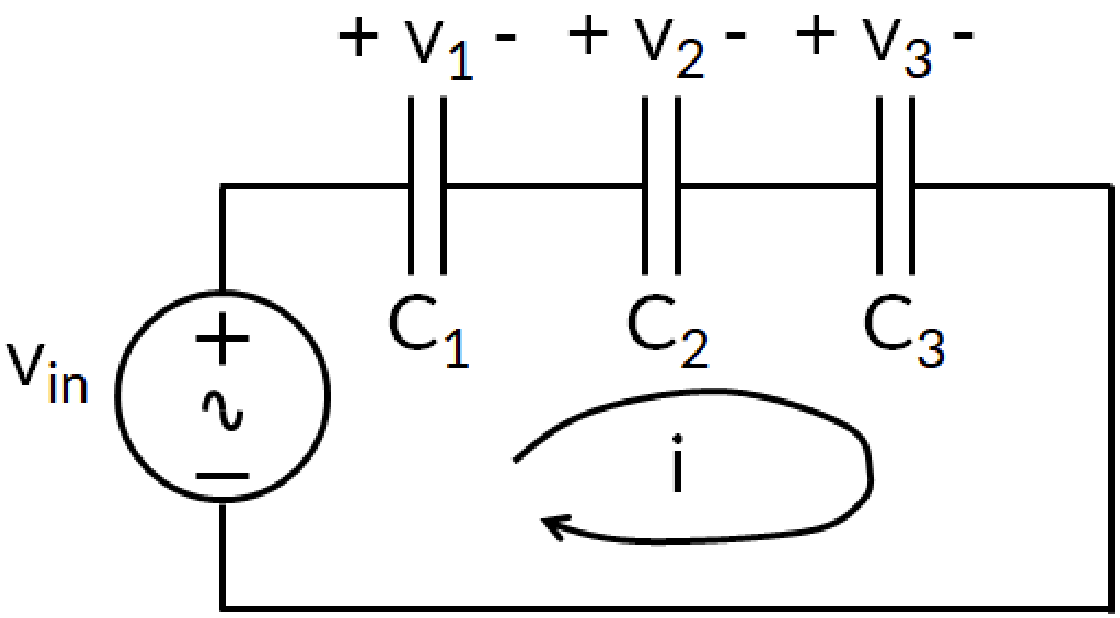
Substituting back into KVL

$$\frac{1}{C_{eq}}\int_{t_0}^{t_1} idt = \frac{1}{C_1}\int_{t_0}^{t_1} idt + \frac{1}{C_2}\int_{t_0}^{t_1} idt + \frac{1}{C_3}\int_{t_0}^{t_1} idt$$

Hence,

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

$$C_{eq} = \left(\sum_{s=1}^{n} C_s^{-1}\right)^{-1}$$



Charging a capacitor

- One of the functions of capacitor is storing charge (and thus energy).
- Capacitor has an ability to store charge when a potential difference is applied across the capacitor plates.
- Energy is stored in the electric field between positive and negative plates.
- When a voltage is applied across a capacitor, current flows into the capacitor plates and develops a potential difference across the capacitor.
- With time, the potential difference between the battery and the capacitor become smaller and the flow rate of electrons (thus current flow) reduces .
- The charging process continues until the capacitor becomes fully charged.
- The charging current follows an exponential curve.

Time Constant

- The rate of charging is determined by the charging equation determined by the RC constant in the exponential term.

$$v_C = \frac{q}{C} = V\left(1 - e^{-\frac{t}{RC}}\right)$$

- The term RC is termed the time constant (mostly RC time constant) since it affects the rate of charge.

- Mathematically, this is the time taken for the capacitor to reach 0.632 of the fully charged value.

- According to the charging equation, theoretically, capacitors takes infinite time to charge completely.

- For all practical purposes, it is assumed that a capacitor can be charged completely in only five times of the time constant, meaning the capacitor is said fully charged after 5×RC.

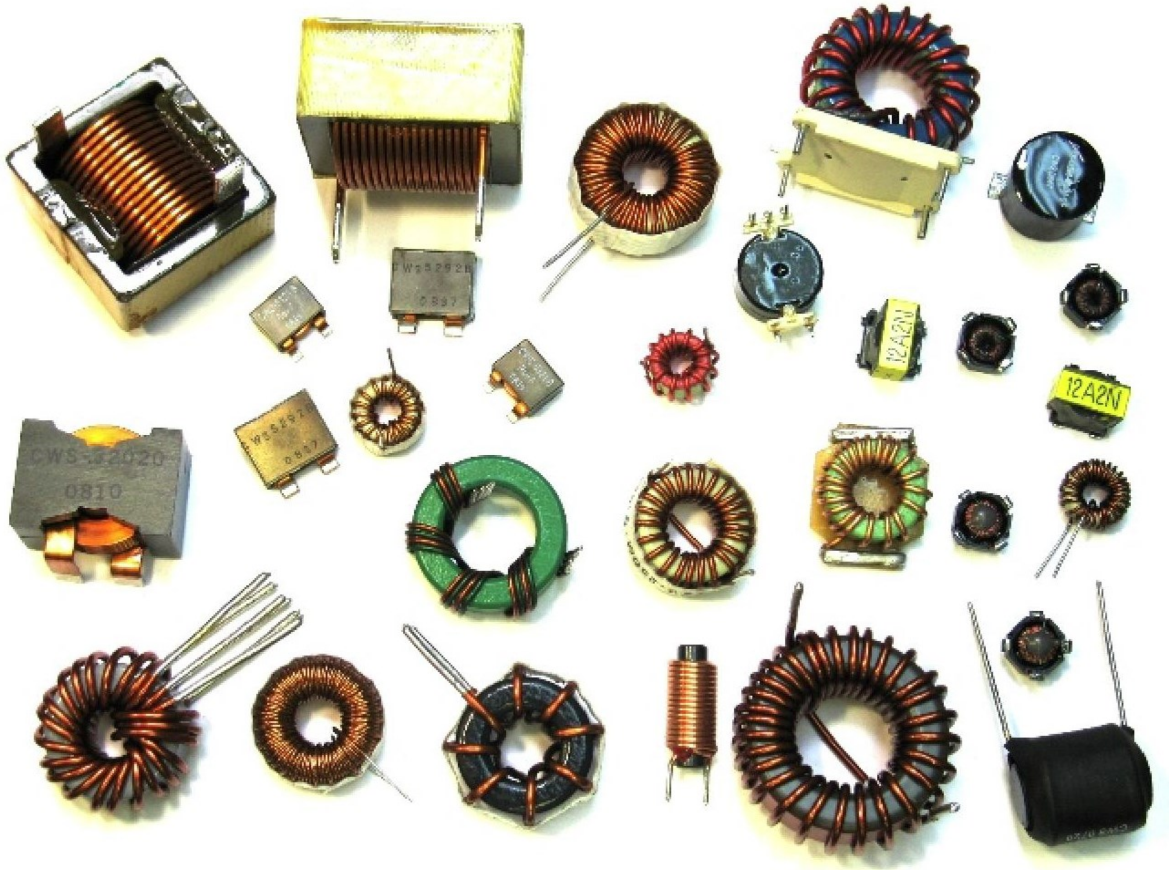
- After 5 time constant, q, Vc and current will be over 99% ($1-e^{-5} = 0.9932$) to their final values.

Inductors

- Inductor stores energy in a magnetic field created by an electric current flowing through it.
- Inductor opposes change (chokes) in current flowing through a conductor.
- Current through an inductor is continuous; voltage can be discontinuous.

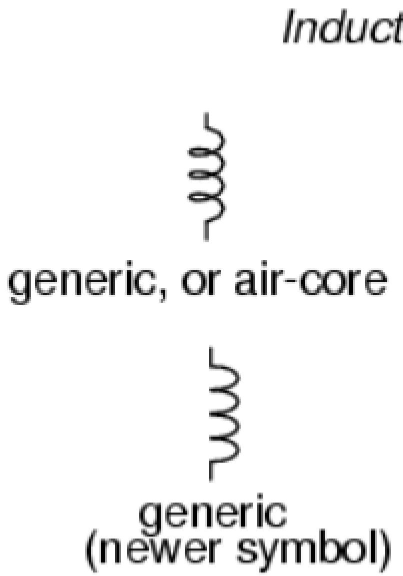
Structure

- Generally formed by a coil of conducting wire
- Conducting wire is usually wrapped around a solid core.
- In the absence of a core, the inductor is said to have an ‘air core’.

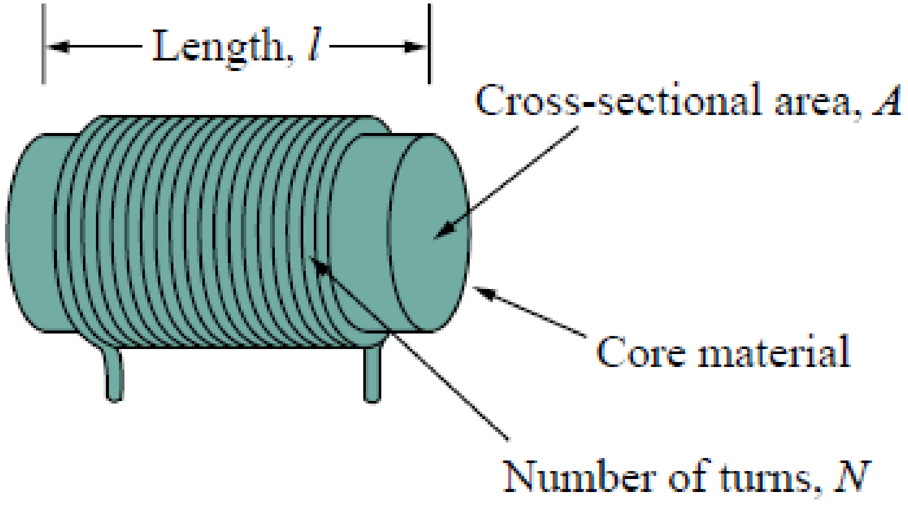


Inductors…Continued

Circuit Symbols



Calculations of L



$$L = \frac{N^2 \mu A}{l} = \frac{N^2 \mu_0 \mu_r A}{l}$$

N is the number of turns of wire
A is the cross-sectional area of the toroid in m^2.
μ_r is the relative permeability of the core material
μ_o is the vacuum permeability ($4\pi \times 10^{-7}$ H/m)
l is the length of the wire used to wrap the toroid in meters

Properties of an Inductor

- Inductor acts like a short circuit in steady state.

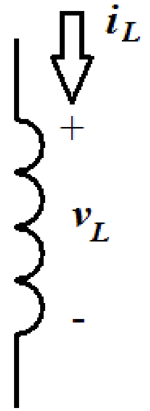
- Current through an inductor must be continuous, meaning there are no abrupt changes to the current but there can be abrupt changes in the voltage across an inductor.

- No energy or power is dissipated by an ideal inductor. Ideal inductor absorbs energy or power from the circuit when storing energy and restores energy into circuit while discharging

Sign Convention

- When current flows into the positive side of the voltage across the inductor, the current is positive, and the inductor is dissipating power

- When an inductor releases energy back into the circuit, the sign of the current is negative.



Current Voltage Relationship

- the voltage across the inductor is directly proportional to the time rate of change of the current.

$$v_L = L\frac{di}{dt}$$

$$i_L = \frac{1}{L}\int_{t_0}^{t_1} v_L\, dt$$

Current Voltage Relationship

Let $I_L = I_0 \sin(\omega t)$

$$v_L = L\frac{d}{dt}\left(I_0 \sin(\omega t)\right)$$

$$v_L = \omega L I_0 cos(\omega t)$$

$$= \omega L I_0 sin\left(\omega t + \frac{\pi}{2}\right)$$

- Voltage across inductor leads current through inductor by $\frac{\pi}{2}$

Let $I_L = I_0 e^{j\omega t}$ $\qquad v_L = L\frac{di}{dt}$

$$v_L = L\frac{d}{dt}\left(I_0 e^{j\omega t}\right) = j\omega L I_0 e^{j\omega t}$$

$$v_L = j\omega L i_L$$

Comparing the equation (similar to that of V= IR)

$$j\omega L = jX_L = Z_L$$

X_L is called as reluctance (in ohm) of an inductor

– When $\omega = 0$, $X_L = 0$, means reactance is zero and inductor behaves like a short at DC

– When $\omega = \infty$, $X_L = \infty$, means inductor behaves like an open circuit at higher frequencies

– Frequency dependent electrical behavior of inductance on circuit

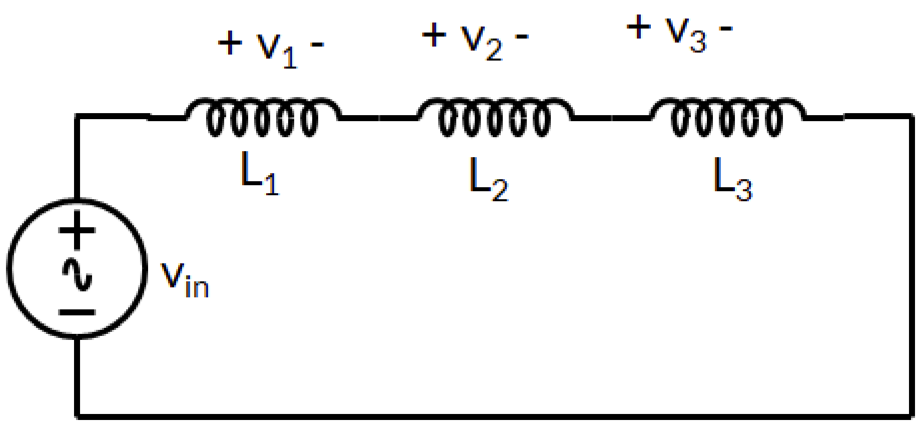
Power and Energy

$$p_L = v_L i_L = L i_L \frac{di_L}{dt}$$

$$w_L = \int_{-\infty}^{t} p_L dt = \int_{-\infty}^{t} L \frac{di_L}{dt} i_L dt = L \int_{-\infty}^{t} i_L d\, i_L = \frac{1}{2} L i_L^2$$

Inductors in Series

- Consider inductors connected in a series configuration as shown in the circuit



- Applying KVL

$$v = v_1 + v_2 + v_3$$

- Noting the relation between voltage across and current through inductor,

$$v_1 = L_1 \frac{di}{dt} \qquad v_2 = L_2 \frac{di}{dt} \qquad v_3 = L_3 \frac{di}{dt}$$

- If L_{eq} is the total inductance of the circuit, then

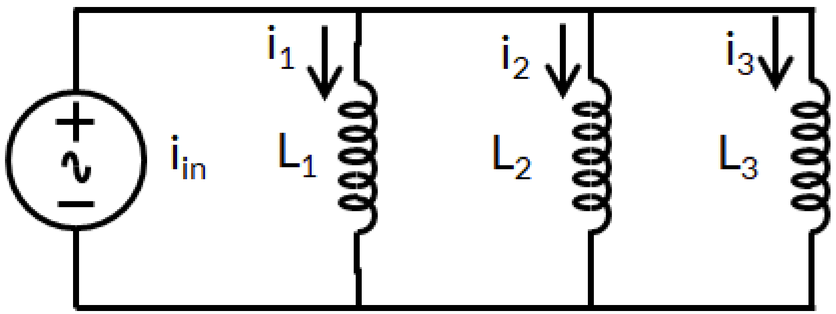
$$v_{in} = L_{eq} \frac{di}{dt} = L_1 \frac{di}{dt} + L_2 \frac{di}{dt} + L_3 \frac{di}{dt}$$

$$L_{eq} = L_1 + L_2 + L_3$$

$$L_{eq} = \sum_{s=1}^{N} L_s$$

Inductors in Parallel

- Consider inductors connected in a parallel configuration as shown in the circuit



- Applying KCL

$$i_{in} = i_1 + i_2 + i_3$$

- Noting the relation between voltage across the inductor and current through inductor

$$i_1 = \frac{1}{L_1}\int_{t_0}^{t_1} vdt \qquad i_2 = \frac{1}{L_2}\int_{t_0}^{t_1} vdt \qquad i_3 = \frac{1}{L_3}\int_{t_0}^{t_1} vdt$$

- If Leq is the total inductance of the circuit, then

$$i_{in} = \frac{1}{L_{eq}}\int_{t_0}^{t_1} vdt = \frac{1}{L_1}\int_{t_0}^{t_1} vdt + \frac{1}{L_2}\int_{t_0}^{t_1} vdt + \frac{1}{L_3}\int_{t_0}^{t_1} vdt$$

$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}$$

$$L_{eq} = \left(\sum_{p=1}^{N} \frac{1}{L_p}\right)^{-1}$$

Passive Filters

Filter Applications

- Filters are specifically used to
  - remove unwanted frequency components from the signal
  - to enhance wanted ones
  - or both.

- Filters are essential building blocks in many systems,
  - Example: used in communication and instrumentation systems

- A common need for filter circuits is in high-performance stereo systems, where certain ranges of audio frequencies need to be amplified or suppressed for best sound quality and power efficiency

Filter Characteristics

- Filter is an electrical network that modifies the amplitude and phase characteristics of a signal with respect to frequency

- In electronic systems, filters are useful in emphasizing signals in certain frequency ranges and reject signals in other frequency ranges.

Different types of Filters

- **Low-pass filter**: low frequencies are passed; high frequencies are attenuated.
- **High-pass filter**: high frequencies are passed; low frequencies are attenuated.
- **Band-pass filter**: only frequencies in a frequency band are passed.
- **Band-stop filter or band-reject filter**: only frequencies in a frequency band are attenuated